

Helicity-1/2 Mode as a Probe of Interactions of Massive Rarita-Schwinger Field

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Abstract

We consider electromagnetic and gravitational interactions of a massive Rarita-Schwinger field. Stückelberg analysis of the system, when coupled to electromagnetism in flat space or to gravity, reveals in either case that the effective field theory has a model-independent upper bound on its UV cutoff, which is finite but parametrically larger than the particle's mass. It is the helicity-1/2 mode that becomes strongly coupled at the cutoff scale. If the interactions are inconsistent, the same mode becomes a telltale evidence of pathologies. Alternatively, consistent interactions are those that propagate this mode within the light cone. Studying its dynamics not only sheds light on the Velo-Zwanziger acausality, but also elucidates why supergravity and other known consistent models are pathology-free.

1 Introduction

The Rarita-Schwinger field carries a spin-3/2 representation of the Poincaré group, whose non-interacting massive theory is described by the following Lagrangian [1]:

$$\mathcal{L}_{\text{free}} = -i\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - im\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu, \quad (1)$$

m being the mass¹. The Dirac equation: $(\not{\partial} - m)\psi_\mu = 0$, along with the correct constraints: $\partial^\mu \psi_\mu = \gamma^\mu \psi_\mu = 0$, can easily be reproduced from the Lagrangian equations of motion. The degrees of freedom count works as follows. The vector-spinor ψ_μ contains $4 \times 4 = 16$ components. The transversality and γ -tracelessness constraints each removes 4 of them, while the Dirac equation removes 4 more, leaving us only with 4, which is indeed the number of physical polarizations of a massive spin-3/2 particle.

When interactions are turned on, as noticed by various authors [2, 3, 4], the theory is generically fraught with inconsistencies even at the classical level², despite the fact that one starts from a Lagrangian as per suggestions made in [6]. The interacting theory may fail to reproduce the necessary constraints that forbid propagating unphysical modes or may give rise to the Velo-Zwanziger acausality [2], i.e. allow faster-than-light speed for the physical modes. Addition of non-minimal terms and/or of new dynamical fields may come as a rescue. For example, the Lagrangian proposed in [7] incorporates appropriate non-minimal terms that propagate causally only the physical modes of a massive spin-3/2 in a constant external electromagnetic (EM) background. A more well-known example is $\mathcal{N} = 2$ (broken) supergravity [8, 9], which contains a massive gravitino that propagates consistently, even when the cosmological constant is set to zero, given that it has a charge, $e = \frac{1}{\sqrt{2}}(m/M_P)$, under the graviphoton [10]. Here causality is preserved by the presence of *both* EM and gravity, along with non-minimal terms.

The pathologies arising in an interacting theory are due to a simple fact: the kinetic part of the free theory (1) enjoys a gauge invariance, and the zero modes may acquire non-vanishing but non-canonical kinetic terms in the presence of interactions. The best way of understanding these issues is the Stückelberg formalism, which was employed in the context of massive spin 2, for example, in [11, 12]. To understand this formalism, let us notice that in the *massive* theory (1) gauge invariance can be restored by introducing a spin-1/2 (Stückelberg) field χ through the field redefinition:

$$\psi_\mu \rightarrow \psi'_\mu = \psi_\mu - \frac{1}{m} \partial_\mu \chi. \quad (2)$$

¹Our conventions are that the metric is mostly positive, the Clifford algebra is $\{\gamma^\mu, \gamma^\nu\} = +2g^{\mu\nu}$, $\gamma^{\mu\dagger} = \eta^{\mu\mu} \gamma^\mu$, $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^{\mu_1\cdots\mu_n} = \frac{1}{n!} \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n} + \text{antisymmetrization}$. The Dirac adjoint is defined as $\bar{\psi}_\mu = \psi_\mu^\dagger \gamma^0$. The totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ is normalized as $\epsilon_{0123} = +1$.

²Pathologies at the quantum level were noticed much earlier in [5], where it was shown that canonical commutators may become ill-defined in an interacting theory.

Now the Lagrangian is manifestly invariant under the Stückelberg symmetry:

$$\delta\psi_\mu = \partial_\mu\lambda, \quad \delta\chi = m\lambda, \quad (3)$$

where λ is a fermionic gauge parameter. Note that when the field redefinition (2) is implemented, potentially bad higher-derivative terms in χ are killed by the antisymmetry of $\gamma^{\mu\nu}$. This is a slick way of understanding the structure of the mass term in (1).

The Stückelberg field is mere redundancy since one can always choose a gauge in which $\chi = 0$, as in Lagrangian (1). The unitary gauge, however, obscures the subtleties associated with an interacting theory, and is therefore not particularly illuminating when interactions are present. On the other hand, as we will see, the intricacies become rather transparent in a different, judiciously chosen, gauge that instead renders the kinetic operators diagonal. For an interacting theory, the latter gauge choice enables one to assign canonical dimensions to potential non-renormalizable operators.

The organization of this paper is as follows. In the Section 2 we consider minimal EM and gravitational couplings of a massive Rarita-Schwinger field, and show that each theory possesses an intrinsic finite UV cutoff, which can be improved neither by field redefinitions nor by addition of non-minimal terms. In Section 3 we perform Stückelberg analysis of various (in)consistent Lagrangians that attempt to describe interactions of a massive spin-3/2 field. In particular, Section 3.1 considers minimal EM coupling and reproduces the Velo-Zwanziger result [2], while Section 3.2 sheds a new light on why the non-minimal Lagrangian presented in [7] is consistent. Section 3.3 reconfirms that minimal gravitational coupling is pathology-free in arbitrary Einstein spaces [13], and finally Section 3.4 analyzes the consistency of $\mathcal{N} = 2$ (broken) supergravity [8, 9, 10]. We conclude with some remarks in Section 4.

2 Ultraviolet Cutoff

Local Lagrangians describing interactions of a massive spin-3/2 field do not have smooth massless limit. Because the free part of the Lagrangian acquires a gauge invariance in this limit, propagators of the massive theory become singular, so that scattering amplitudes diverge. Notice, however, that if we introduce minimal coupling (to EM or gravity) in the Rarita-Schwinger action (1), no inverse powers of the mass appear in the resulting Lagrangian. Thus the massless singularity is not at all obvious in the unitary gauge.

The Stückelberg formalism, on the other hand, focuses precisely on the gauge modes responsible for bad high energy behavior. One can “invent” the Stückelberg symmetry and then exploit it to make a judicious covariant gauge fixing such that the propagators acquire smooth massless limit. In this gauge one will end up having explicit dependence

on inverse powers of the mass in the form of non-renormalizable interaction terms that involve the Stückelberg field χ . The cutoff scale can be read off from the most divergent terms in the Lagrangian – the terms that survive in an appropriate scaling limit of zero mass and zero coupling.

2.1 EM Coupling in Flat Space

First we consider EM coupling in flat space, and show that the theory has an upper bound on its UV cutoff³. When minimally coupled to a $U(1)$ gauge field, the Stückelberg invariant Lagrangian for a massive Rarita-Schwinger field reads

$$\mathcal{L}_{\text{em}} = -i \left(\bar{\psi}_\mu - \frac{1}{m} \bar{\chi} \overleftarrow{D}_\mu \right) (\gamma^{\mu\nu\rho} D_\nu + m \gamma^{\mu\rho}) \left(\psi_\rho - \frac{1}{m} D_\rho \chi \right) - \frac{1}{4} F_{\mu\nu}^2, \quad (4)$$

which has the manifest gauged Stückelberg symmetry:

$$\delta\psi_\mu = D_\mu \lambda, \quad \delta\chi = m\lambda, \quad (5)$$

where the covariant derivatives obey $[D_\mu, D_\nu] = ieF_{\mu\nu}$. More explicitly,

$$\mathcal{L}_{\text{em}} = \mathcal{L}_{3/2} + \mathcal{L}_{\text{mix}} + \mathcal{L}_{\text{int}} - \frac{1}{4} F_{\mu\nu}^2, \quad (6)$$

where $\mathcal{L}_{3/2}$ involves only the helicity-3/2 mode, \mathcal{L}_{mix} is the kinetic mixing between the two modes, and \mathcal{L}_{int} are non-renormalizable interaction terms, respectively given as

$$\mathcal{L}_{3/2} = -i \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - im \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu, \quad (7)$$

$$\mathcal{L}_{\text{mix}} = i(\bar{\psi}_\mu \gamma^{\mu\nu} D_\nu \chi + \bar{\chi} \overleftarrow{D}_\mu \gamma^{\mu\nu} \psi_\nu), \quad (8)$$

$$\mathcal{L}_{\text{int}} = \frac{e}{2m} F_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu\rho} \psi_\rho - \bar{\psi}_\rho \gamma^{\mu\nu\rho} \chi - \bar{\chi} \gamma^{\mu\nu} \chi) - \frac{e}{2m^2} F_{\mu\nu} \bar{\chi} \gamma^{\mu\nu\rho} D_\rho \chi. \quad (9)$$

The kinetic mixing can be removed by a field redefinition, namely

$$\psi_\mu \rightarrow \psi_\mu + \frac{1}{2} \gamma_\mu \chi, \quad (10)$$

which, at the same time, produces a kinetic term for χ as well as mass mixing. Now we can add the following gauge-fixing term to the Lagrangian:

$$\mathcal{L}_{\text{gf}} = i \bar{\psi}_\mu (\gamma^{\mu\nu} \gamma^\rho - \gamma^\mu \eta^{\nu\rho}) D_\nu \psi_\rho + im \bar{\psi}_\mu \gamma^\mu \gamma^\nu \psi_\nu + \frac{3}{2} im (\bar{\psi}_\mu \gamma^\mu \chi - \bar{\chi} \gamma^\mu \psi_\mu - \bar{\chi} \chi), \quad (11)$$

which renders the propagators smooth in the massless limit, thanks to the identity

$$\gamma^{\mu\nu\rho} = \gamma^{\mu\nu} \gamma^\rho + \eta^{\mu\rho} \gamma^\nu - \eta^{\nu\rho} \gamma^\mu. \quad (12)$$

³This was originally considered in [14]. Here we reconsider it, with a more refined analysis, for the sake of completeness. The analysis will also be useful for the latter parts of the paper.

The same removes the mass mixing as well, finally giving

$$\begin{aligned}\mathcal{L}_{\text{em}} = & -i\bar{\psi}_\mu(\not{D} - m)\psi^\mu - \frac{3}{2}i\bar{\chi}(\not{D} - m)\chi - \frac{1}{4}F_{\mu\nu}^2 \\ & + \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}\gamma^{\mu\nu}\chi) - \frac{e}{2m^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi). \quad (13)\end{aligned}$$

For $e \ll 1$, the most dangerous terms in the high energy limit are the dimension-6 operators. Note that the degree of divergence does not improve by the addition of non-minimal terms, since any such operator is necessarily irrelevant. Even a dipole term,

$$\mathcal{L}_{\text{dipole}} = \frac{ea}{m}F^{\mu\nu}\bar{\psi}_\mu\psi_\nu \rightarrow \frac{ea}{m}F^{\mu\nu}\left(\bar{\psi}_\mu - \frac{1}{2}\bar{\chi}\gamma_\mu - \frac{1}{m}\bar{\chi}\overleftarrow{D}_\mu\right)\left(\psi_\nu + \frac{1}{2}\gamma_\nu\chi - \frac{1}{m}D_\nu\chi\right), \quad (14)$$

introduces, among others, equally bad but new dimension-6 operators that involve both the helicities. Clearly, higher multipole operators will worsen the degree of divergence⁴. Now one can take the scaling limit: $m \rightarrow 0$ and $e \rightarrow 0$, such that $m^2/e \equiv \Lambda_{\text{em}}^2 = \text{constant}$. The Lagrangian then reduces, after the rescaling $\chi \rightarrow \sqrt{\frac{2}{3}}\chi$, to

$$\mathcal{L}_{\text{em}} \rightarrow -i\bar{\psi}_\mu \not{\partial}\psi^\mu - i\bar{\chi} \not{\partial}\chi - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{3\Lambda_{\text{em}}^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\partial_\rho\chi). \quad (15)$$

Notice, however, that the non-renormalizable operators in (15) are all proportional to the equations of motion, up to total derivatives. Indeed one can use identity (12) to write

$$F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\partial_\rho\chi) = \frac{1}{2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu} \not{\partial}\chi - \bar{\chi}\overleftarrow{\not{\partial}}\gamma^{\mu\nu}\chi) - \partial_\mu F^{\mu\nu}(\bar{\chi}\gamma_\nu\chi). \quad (16)$$

Therefore, one can eliminate them by appropriate field redefinitions of χ and A_μ , namely

$$\chi \rightarrow \chi + \frac{i}{6\Lambda_{\text{em}}^2}F_{\mu\nu}\gamma^{\mu\nu}\chi, \quad A_\mu \rightarrow A_\mu - \frac{1}{3\Lambda_{\text{em}}^2}\bar{\chi}\gamma_\mu\chi, \quad (17)$$

as canceling contributions come from the helicity-1/2 and photon kinetic terms. The price one has to pay is that new non-renormalizable operators of dimensions 8, 10 and 12 show up, all with various negative powers of the scale Λ_{em} . Can we add local counter-terms to the original action, which eliminate all these operators up to total derivatives, and introduce only such new terms as vanish in the above scaling limit? A positive answer would mean that one may improve the degree of divergence of the minimally coupled theory by field redefinitions plus addition of local counter-terms. To see that this is not the case, let us consider the dimension-8 operator $(\bar{\chi}\gamma^\mu\chi)\square(\bar{\chi}\gamma_\mu\chi)$, which comes from the photon-field redefinition acting on the last term of (16). It is neither proportional to the

⁴We emphasize that here we are only up to improving the degree of divergence, as we are looking for a theoretical upper bound on the cutoff scale that no theory can beat. No way do we mean that non-minimal terms are forbidden. In fact, they do appear in consistent models, e.g. supergravity. But then the theory will have a cutoff which is simply lower than the upper bound we are trying to find.

equations of motion nor does it contain the EM field strength. Without worsening the degree of divergence, such operators may only be produced by 4-Fermi-like local counter-terms, which in the unitary gauge look like $(e^2/m^2)\bar{\psi}\psi\bar{\psi}\psi$. More explicitly,

$$\begin{aligned} \mathcal{L}_{\text{c.t.}} \rightarrow & b \left(\frac{e}{m} \right)^2 \left(\bar{\psi}_\mu - \frac{1}{2} \bar{\chi} \gamma_\mu - \frac{1}{m} \bar{\chi} \overleftarrow{D}_\mu \right) \gamma^{\mu\nu\rho\sigma} \left(\psi_\nu + \frac{1}{2} \gamma_\nu \chi - \frac{1}{m} D_\nu \chi \right) \\ & \times \left(\bar{\psi}_\rho - \frac{1}{2} \bar{\chi} \gamma_\rho - \frac{1}{m} \bar{\chi} \overleftarrow{D}_\rho \right) \left(\psi_\sigma + \frac{1}{2} \gamma_\sigma \chi - \frac{1}{m} D_\sigma \chi \right) + \dots, \end{aligned} \quad (18)$$

where the $\gamma^{\mu\nu\rho\sigma}$ plays the essential role of killing the more dangerous operators. However, such counter-terms produce, on top of those that we want to eliminate, new dimension-8 operators, involving both helicities, that survive in the scaling limit.

Thus the effective field theory of a massive Rarita-Schwinger field interacting with EM in flat space has a finite intrinsic upper bound on its cutoff:

$$\Lambda_{\text{em}} = \frac{m}{\sqrt{e}}, \quad (19)$$

which is parametrically larger than m . As seen from (15), the breakdown of the effective action is due to the helicity-1/2 mode χ that becomes strongly coupled at high energies.

2.2 Gravitational Coupling

The Stückelberg invariant action for a massive spin 3/2, minimally coupled to gravity, is

$$\mathcal{L}_g = -i\sqrt{-g} \left(\bar{\psi}_\mu - \frac{1}{m} \bar{\chi} \overleftarrow{\nabla}_\mu \right) (\gamma^{\mu\nu\rho} \nabla_\nu + m \gamma^{\mu\rho}) \left(\psi_\rho - \frac{1}{m} \nabla_\rho \chi \right) + \frac{1}{2} M_{\text{P}}^2 \sqrt{-g} R. \quad (20)$$

Here the commutator of covariant derivatives acts on different modes as:

$$[\nabla_\mu, \nabla_\nu] \psi_\rho = -R_{\mu\nu\rho}{}^\sigma \psi_\sigma + \frac{1}{4} R_{\mu\nu\alpha\beta} \gamma^{\alpha\beta} \psi_\rho, \quad (21)$$

$$[\nabla_\mu, \nabla_\nu] \chi = \frac{1}{4} R_{\mu\nu\alpha\beta} \gamma^{\alpha\beta} \chi. \quad (22)$$

One can work out the Lagrangian (20) to write

$$\mathcal{L}_g = \mathcal{L}_{3/2} + \mathcal{L}_{\text{mix}} + \mathcal{L}_{\text{int}} + \frac{1}{2} M_{\text{P}}^2 \sqrt{-g} R, \quad (23)$$

where $\mathcal{L}_{3/2}$ and \mathcal{L}_{mix} are the gravitational counterparts of those given by Eqs. (7) and (8) respectively, while \mathcal{L}_{int} are the non-renormalizable interactions. The latter can be computed explicitly, given Eqs. (21)-(22), the Bianchi identity: $R_{[\mu\nu\alpha]\beta} = 0$, and various γ -matrix identities. The following ones are particularly useful:

$$\gamma^{\mu\nu\rho} \gamma^{\alpha\beta} R_{\mu\nu\alpha\beta} (\psi_\rho, \nabla_\rho \chi) = 4G^{\mu\nu} \gamma_\mu (\psi_\nu, \nabla_\nu \chi), \quad \gamma^{\mu\nu} \gamma^{\alpha\beta} R_{\mu\nu\alpha\beta} = -2R, \quad (24)$$

where $G^{\mu\nu}$ is the Einstein tensor. The result is

$$\mathcal{L}_{\text{int}} = -\frac{i}{2m} \sqrt{-g} \left[G^{\mu\nu} (\bar{\chi} \gamma_\mu \psi_\nu - \bar{\psi}_\mu \gamma_\nu \chi) + \frac{1}{2} \bar{\chi} R \chi \right] + \frac{i}{2m^2} \sqrt{-g} G^{\mu\nu} \bar{\chi} \gamma_\mu \nabla_\nu \chi. \quad (25)$$

The field redefinition that eliminates the kinetic mixing is the same as (10), while the desired gauge-fixing term is just the gravitational counterpart of (11). One is left with

$$\begin{aligned} \mathcal{L}_g = & -i\sqrt{-g} \left[\bar{\psi}_\mu (\not{\nabla} - m) \psi^\mu + \frac{3}{2} \bar{\chi} (\not{\nabla} - m) \chi \right] + \frac{1}{2} M_{\text{P}}^2 \sqrt{-g} R \\ & - \frac{i}{2m} \sqrt{-g} \left[G^{\mu\nu} (\bar{\chi} \gamma_\mu \psi_\nu - \bar{\psi}_\mu \gamma_\nu \chi) - \frac{1}{2} \bar{\chi} R \chi - \frac{1}{m} G^{\mu\nu} \bar{\chi} \gamma_\mu \nabla_\nu \chi \right]. \end{aligned} \quad (26)$$

Before assigning canonical dimensions to various operators, we must canonically normalize the graviton field $h_{\mu\nu}$, so that it has mass dimension one:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\text{P}}} h_{\mu\nu}. \quad (27)$$

We take $m \ll M_{\text{P}}$, which is essential for a sensible effective field theory to exist. We see that in the high energy limit the most dangerous terms are the dimension-7 operators contained in $G^{\mu\nu} \bar{\chi} \gamma_\mu \nabla_\nu \chi$, which are $\chi - h - \chi$ vertices. Because non-minimal interactions show up with Planck-mass suppression in the unitary gauge, they can contribute only less divergent terms to the Lagrangian (26). Thus they are harmless, but they do not improve the degree of divergence either. In the scaling limit: $m \rightarrow 0$ and $M_{\text{P}} \rightarrow \infty$, such that $m^2 M_{\text{P}} \equiv \Lambda_{\text{g}}^3 = \text{constant}$, and with the rescaling $\chi \rightarrow \sqrt{\frac{2}{3}} \chi$, we are left with

$$\mathcal{L}_g \rightarrow -i\bar{\psi}_\mu \not{\partial} \psi^\mu - i\bar{\chi} \not{\partial} \chi - h_{\mu\nu} \mathcal{G}^{\mu\nu} + \frac{i}{3\Lambda_{\text{g}}^3} \mathcal{G}^{\mu\nu} \bar{\chi} \gamma_\mu \partial_\nu \chi. \quad (28)$$

Here $\mathcal{G}^{\mu\nu} \equiv (\mathcal{E} \cdot h)^{\mu\nu}$ is the linearized Einstein tensor, and

$$\mathcal{E}^{\mu\nu\alpha\beta} = \frac{1}{2} \left[(\eta^{\mu\nu} \eta^{\alpha\beta} - \eta^{\mu\nu, \alpha\beta}) \square - \eta^{\mu\nu} \partial^\alpha \partial^\beta - \eta^{\alpha\beta} \partial^\mu \partial^\nu + \eta^{\mu(\alpha} \partial^{\beta)} \partial^\nu + \eta^{\nu(\alpha} \partial^{\beta)} \partial^\mu \right], \quad (29)$$

so that $-h_{\mu\nu} \mathcal{G}^{\mu\nu}$ is the kinetic term for the canonically normalized graviton $h_{\mu\nu}$. It is clear that the dimension-7 operator in (28) can be eliminated by the field redefinition:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \frac{i}{6\Lambda_{\text{g}}^3} \bar{\chi} \gamma_{(\mu} \partial_{\nu)} \chi. \quad (30)$$

But this will leave us with the following dimension-10 operator, quartic in χ :

$$\mathcal{L}_{\text{dim-10}} = \frac{1}{36\Lambda_{\text{g}}^6} (\bar{\chi} \gamma_\mu \partial_\nu \chi) \mathcal{E}^{\mu\nu\alpha\beta} (\bar{\chi} \gamma_\alpha \partial_\beta \chi). \quad (31)$$

This operator contains pieces that cannot be removed by field redefinitions, but may be canceled, up to total derivatives, by addition of local counter-terms. Indeed 4-Fermi

terms in the unitary gauge produce, after the replacement $\psi_\mu \rightarrow \psi_\mu + \frac{1}{2}\gamma_\mu\chi - (1/m)\partial_\mu\chi$, dimension-10 operators quartic in χ , which are invariant under the shift of χ by a constant spinor. Cancellation requires that we consider *all* of (31), because *only the entire operator* enjoys this shift symmetry. Potentially interesting unitary-gauge counter-terms must look like: $M_{\text{P}}^{-2}(\bar{\psi}_\rho\gamma_\mu\psi_\nu)\mathcal{A}^{\mu\nu\alpha\beta,\rho\sigma}(\bar{\psi}_\sigma\gamma_\alpha\psi_\beta)$, where $\mathcal{A}^{\mu\nu\alpha\beta,\rho\sigma}$ is a dimensionless tensor. In order to cancel (31), it is required that $\partial_\rho\partial_\sigma\mathcal{A}^{\mu\nu\alpha\beta,\rho\sigma}$ be proportional to $\mathcal{E}^{\mu\nu\alpha\beta}$. Therefore, the tensor \mathcal{A} cannot be antisymmetric in the last two indices. However, this also means that one must have operators of new kinds, like $(\bar{\chi}\overleftarrow{\partial}_\rho\overleftarrow{\partial}_\sigma\gamma_\mu\partial_\nu\chi)\mathcal{A}^{\mu\nu\alpha\beta,\rho\sigma}(\bar{\chi}\gamma_\alpha\partial_\beta\chi)$, which are not total derivatives. The conclusion is that the degree of divergence of the minimal theory cannot be improved by field redefinitions plus addition of local counter-terms.

Thus we have found an upper bound on the UV cutoff of the effective theory describing a gravitationally interacting massive spin-3/2 field:

$$\Lambda_g = \sqrt[3]{m^2 M_{\text{P}}} . \quad (32)$$

This is finite but parametrically larger than the mass. Eq. (28) reveals that it is the helicity-1/2 mode, once again, that exhibits strong coupling around the cutoff scale.

3 Interacting Theories of Rarita-Schwinger Field

Now we will consider various (in)consistent models of an interacting massive spin 3/2, and analyze them through Stückelberg formalism. As we already know, when interactions are present, the helicity-1/2 mode generally acquires non-standard kinetic terms. In inconsistent theories this mode may move faster than light or even cease to propagate. Consistency of interacting theories crucially relies on having a pathology-free helicity-1/2 sector. Conversely, by ensuring that this mode does not exhibit pathological behavior, we can (re)derive conditions that render a theory consistent.

3.1 Minimal EM Interaction

This has already been considered in Section 2.1, and we recall from Eq. (13) that the Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{\text{em}} = & -i\bar{\psi}_\mu(\not{D} - m)\psi^\mu - \frac{3}{2}i\bar{\chi}(\not{D} - m)\chi - \frac{1}{4}F_{\mu\nu}^2 \\ & + \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}\gamma^{\mu\nu}\chi) - \frac{e}{2m^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi). \end{aligned} \quad (33)$$

It is manifest that the helicity-3/2 sector enjoys a healthy kinetic term. On the other hand, the χ -sector is tricky, because in an external EM background the last term in (33) will act like a kinetic operator. Let us write down the equations of motion for χ :

$$-i\not{\partial}\chi - \frac{1}{2}\alpha\gamma^{\mu\nu\rho}F_{\mu\nu}\partial_\rho\chi + (\text{lower-derivative terms}) = 0, \quad \alpha \equiv \frac{2}{3}e/m^2. \quad (34)$$

Now we will use the method of characteristic determinant [2] to investigate whether this system allows propagation outside the light cone. The method consists of determining normal, $n_\mu = (n_0, \vec{n})$, to the characteristic hypersurfaces. We replace ∂_μ with $-in_\mu$ in the highest derivative terms in (34), and then equate to zero the determinant $\Delta(n)$ of the resulting coefficient matrix. The system is hyperbolic, if for any \vec{n} , the algebraic equation $\Delta(n) = 0$ has real solutions for n_0 , and then the ratio $n_0/|\vec{n}|$ gives the maximum wave speed. The required coefficient matrix is given by

$$\begin{aligned} \mathcal{M} &= -\gamma^\mu n_\mu + \frac{i}{2}\alpha \gamma^{\mu\nu\rho} F_{\mu\nu} n_\rho \\ &= -i \begin{pmatrix} \mathbf{0} & -\vec{\sigma} \cdot (\vec{n} + \alpha n_0 \vec{B}) - (n_0 + \alpha \vec{n} \cdot \vec{B}) \\ \vec{\sigma} \cdot (\vec{n} - \alpha n_0 \vec{B}) - (n_0 - \alpha \vec{n} \cdot \vec{B}) & \mathbf{0} \end{pmatrix}. \end{aligned} \quad (35)$$

To compute its determinant let us assume, without loss of generality, that the magnetic field \vec{B} points in the z -direction, and that the 3-vector \vec{n} lies on the zx -plane making an angle θ with \vec{B} . Thus we obtain

$$\Delta(n) \equiv \det \mathcal{M} = [n_0^2 - \vec{n}^2 - \alpha^2 \vec{B}^2 (n_0^2 - \vec{n}^2 \cos^2 \theta)]^2, \quad (36)$$

which vanishes for

$$\frac{n_0}{|\vec{n}|} = \sqrt{\frac{1 - \alpha^2 \vec{B}^2 \cos^2 \theta}{1 - \alpha^2 \vec{B}^2}}. \quad (37)$$

We see that the system ceases to be hyperbolic whenever $\alpha^2 \vec{B}^2$ exceeds unity, i.e. when

$$\vec{B}^2 \geq \left(\frac{3m^2}{2e} \right)^2. \quad (38)$$

Not only that, even an infinitesimal magnetic field will cause superluminal propagation for generic θ . This is the so-called Velo-Zwanziger problem. The pathologies are serious in that they can very well arise when the EM field invariants $\vec{B}^2 - \vec{E}^2$ and $\vec{B} \cdot \vec{E}$ are non-vanishing but small (in the units of m^4/e^2), so that we are far away from the regime of instabilities [15] and the notion of long-lived propagating particles makes sense.

3.2 Consistent Non-Minimal EM Couplings

The Velo-Zwanziger acausality shows up even for the simplest possible interaction setup of a constant external EM background. A wide class of non-minimal models [3] also exhibits the same pathological features. Porrati and Rahman [7] wrote down a non-minimal Lagrangian, which consistently describes a charged massive Rarita-Schwinger field exposed to a constant EM background in flat space. In the unitary gauge it reads [7]:

$$\mathcal{L}_{\text{PR}} = -i\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - im\bar{\psi}_\mu b^{\mu\nu} \psi_\nu, \quad (39)$$

where the antisymmetric tensor $b_{\mu\nu}$ contains “corrections” to $\gamma_{\mu\nu}$, of the form

$$b_{\mu\nu} = \gamma_{\mu\nu} + B_{\mu\nu}^+ + \gamma^\rho C_{\rho[\mu} \gamma_{\nu]}, \quad B_{\mu\nu}^\pm \equiv B_{\mu\nu} \mp i\gamma_5 \tilde{B}_{\mu\nu}, \quad (40)$$

with $\tilde{B}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$. The Lorentz tensor $B_{\mu\nu}$ is antisymmetric, while the Lorentz tensor $C_{\mu\nu}$ is symmetric traceless. They are respectively imaginary and real, as implied by the hermiticity condition, and they both vanish in the limit $F \rightarrow 0$. They are related as [7]:

$$C^{\mu\nu} = -\frac{1}{2}B^{+\mu\rho}B_{\rho}^{-\nu} = -\frac{1}{2}B^{-\mu\rho}B_{\rho}^{+\nu} = -[B^{\mu\rho}B_{\rho}^{\nu} - \frac{1}{4}\eta^{\mu\nu}\text{Tr}(B^2)], \quad (41)$$

while $B_{\mu\nu}$ can be computed from the following relation [7]:

$$B_{\mu\nu} = i(e/m^2)F_{\mu\nu} + \frac{1}{4}\text{Tr}(B^2)B_{\mu\nu} - \frac{1}{4}\text{Tr}(B\tilde{B})\tilde{B}_{\mu\nu}, \quad (42)$$

as a power series in the EM field strength $F_{\mu\nu}$, which is always possible in physically interesting situations, i.e. when the EM field invariants are small.

In what follows we perform Stückelberg analysis of the Lagrangian (39) to reveal that the relations (41) and (42) are precisely those that ensure a healthy helicity-1/2 sector. We can render the Lagrangian Stückelberg invariant as usual, and work out the various terms to arrive at the non-minimal counterpart of Eq. (6):

$$\begin{aligned} \mathcal{L}_{\text{PR}} = & -i\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - im\bar{\psi}_\mu b^{\mu\nu}\psi_\nu + i(\bar{\psi}_\mu b^{\mu\nu}D_\nu\chi + \bar{\chi}\overleftarrow{D}_\mu b^{\mu\nu}\psi_\nu) \\ & + \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi - \bar{\chi}b^{\mu\nu}\chi) - \frac{e}{2m^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi). \end{aligned} \quad (43)$$

As we have already seen in the minimal theory, a redefinition of ψ_μ can eliminate the kinetic mixing. To find such a field redefinition in the present case, let us take note of the following identities that follow from elementary manipulations of γ -matrix algebra:

$$B_{\mu\nu}^+ = -\frac{1}{4}\gamma^\rho \not{B} \gamma_{\rho\mu\nu} = -\frac{1}{4}\gamma_{\mu\nu\rho} \not{B} \gamma^\rho, \quad \not{B} \equiv \gamma^{\mu\nu}B_{\mu\nu}, \quad (44)$$

$$\gamma^\alpha C_{\alpha[\mu} \gamma_{\nu]} = -\frac{1}{2}\gamma_\alpha C^{\alpha\rho} \gamma_{\rho\mu\nu} = -\frac{1}{2}\gamma_{\mu\nu\rho} C^{\rho\alpha} \gamma_\alpha. \quad (45)$$

Given this, it is not difficult to see that the required field redefinition is

$$\psi_\mu \rightarrow \psi_\mu + \frac{1}{2}(\gamma_\mu - \frac{1}{2}\not{B}\gamma_\mu - \gamma^\alpha C_{\alpha\mu})\chi. \quad (46)$$

This, when implemented in (43), will also produce new non-canonical kinetic operators for χ , which add to the already existing troublesome operator $F_{\mu\nu}\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi$. One can also add the gauge-fixing term (11) to make manifest that the helicity-3/2 sector is hyperbolic and causal. The result is the non-minimal counterpart of (13), given by

$$\begin{aligned} \mathcal{L}_{\text{PR}} = & -i\bar{\psi}_\mu(\not{D} - m)\psi^\mu - \frac{3}{2}i\bar{\chi}(\not{D} - m)\chi + \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}b^{\mu\nu}\chi) \\ & + \frac{1}{2}i\bar{\chi}[i(e/m^2)\gamma^{\mu\nu\rho}F_{\mu\nu} + b^{\mu\rho}(\gamma_\mu - \frac{1}{2}\not{B}\gamma_\mu - \gamma^\alpha C_{\alpha\mu}) + 3\gamma^\rho]D_\rho\chi. \end{aligned} \quad (47)$$

The key point is that we have at our disposal two functions of the EM field strength, $B_{\mu\nu}$ and $C_{\mu\nu}$, which could be judiciously chosen so as to render the χ -sector pathology-free. With this end in view, we make the rescaling $\chi \rightarrow \sqrt{\frac{2}{3}}\chi$, and look at the helicity-1/2 kinetic-like operators, which we symbolically write as

$$\mathcal{L}_{\chi,\text{kin}} = -i\bar{\chi}\Gamma^\mu D_\mu\chi. \quad (48)$$

If Γ^μ is proportional to γ^μ with a positive coefficient, the χ -sector will be ghost-free, hyperbolic and causal. The expression for Γ^μ can be simplified to

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu + \frac{1}{3} \left\{ -i(e/m^2)\gamma^{\mu\nu\rho}F_{\nu\rho} + \gamma^{\mu\nu\rho}B_{\nu\rho} + \gamma^\rho C_{\rho\nu} (B^{-\nu\mu} - \frac{1}{2}B^{+\nu\alpha}\gamma^\mu\gamma_\alpha) \right\} \\ &\quad + \frac{1}{3}\gamma_\nu [2C^{\mu\nu} + B^{+\mu\rho}B_\rho^{-\nu}] + \frac{1}{6}\gamma^\nu C_{\nu\rho}C^{\rho\sigma}\gamma^\mu\gamma_\sigma, \end{aligned} \quad (49)$$

thanks to the identities:

$$\gamma^\mu B_{\mu\nu}^+ = \frac{1}{2} \not{B} \gamma_\nu, \quad B_{\mu\nu}^+ \gamma^\nu = \frac{1}{2} \gamma_\mu \not{B}, \quad \gamma_\mu \not{B} \gamma^\mu = 0, \quad \frac{1}{2} (\gamma^\mu \not{B} + \not{B} \gamma^\mu) = \gamma^{\mu\nu\rho} B_{\nu\rho}. \quad (50)$$

In (49) if one sets to zero the symmetric traceless tensor inside the brackets, which is nothing but the choice of relation (41), the entire second line becomes proportional to γ^μ . This is because of the identity

$$B_{\mu\rho}^\pm B^{\pm\rho\nu} = \frac{1}{2} [\text{Tr}(B^2) \pm i\gamma_5 \text{Tr}(B\tilde{B})] \delta_\mu^\nu, \quad (51)$$

which, along with (41), enables one to write

$$\gamma^\nu C_{\nu\rho} C^{\rho\sigma} \gamma^\mu \gamma_\sigma = \frac{1}{4} \gamma^\nu (B_\nu^{+\alpha} B_{\alpha\rho}^- B^{-\rho\lambda} B_\lambda^{+\sigma}) \gamma^\mu \gamma_\sigma = -\frac{1}{8} \left\{ [\text{Tr}(B^2)]^2 + [\text{Tr}(B\tilde{B})]^2 \right\} \gamma^\mu. \quad (52)$$

Moreover, one can use Eqs. (41) and (51), and the definitions of $B_{\mu\nu}^\pm$ and $\tilde{B}_{\mu\nu}$ to write

$$\gamma^\rho C_{\rho\nu} (B^{-\nu\mu} - \frac{1}{2}B^{+\nu\alpha}\gamma^\mu\gamma_\alpha) = -\frac{1}{4}\gamma^{\mu\nu\rho} \left[\text{Tr}(B^2)B_{\nu\rho} - \text{Tr}(B\tilde{B})\tilde{B}_{\nu\rho} \right]. \quad (53)$$

Now in view of Eqs. (41), (52) and (53), the expression for Γ^μ reduces to

$$\begin{aligned} \Gamma^\mu &= \left\{ 1 - \frac{1}{48}[\text{Tr}(B^2)]^2 - \frac{1}{48}[\text{Tr}(B\tilde{B})]^2 \right\} \gamma^\mu \\ &\quad + \frac{1}{3}\gamma^{\mu\nu\rho} \left\{ -i(e/m^2)F_{\nu\rho} + B_{\nu\rho} - \frac{1}{4}\text{Tr}(B^2)B_{\nu\rho} + \frac{1}{4}\text{Tr}(B\tilde{B})\tilde{B}_{\nu\rho} \right\}. \end{aligned} \quad (54)$$

This produces the same kind of helicity-1/2 kinetic terms as the minimally-coupled theory. Clearly, the second line in the above expression will give rise to pathologies unless it is set to zero. Then, consistent propagation of χ requires (42), and we are left with

$$\mathcal{L}_{\chi,\text{kin}} = -i \left\{ 1 - \frac{1}{48}[\text{Tr}(B^2)]^2 - \frac{1}{48}[\text{Tr}(B\tilde{B})]^2 \right\} \bar{\chi} \not{D} \chi. \quad (55)$$

The factor appearing in the kinetic term manifestly depends on the relativistic field invariants in such a way that it is always positive in the regimes of physical interest. Thus the mere requirement of a healthy helicity-1/2 sector recovers the model (39)-(42).

3.3 Minimal Coupling to Gravity

Already considered in Section 2.2, minimal gravitational coupling shows up, interestingly, as one tries to write down consistent models for a massive spin 3/2 in Einstein space [13]. The Lagrangian found in [13] (by using the BRST approach to higher-spin fields) boils down to the minimal Lagrangian in the unitary gauge. It means that if we take the minimally coupled theory with the spin 3/2 as a probe, consistent propagation of the helicity-1/2 mode would require that the Einstein tensor be proportional to the metric.

The consistency of minimal gravitational coupling in Einstein spaces becomes rather obvious in the Stückelberg language. We recall from Eq. (26) that the minimally coupled theory, in $d = 4$ dimensions, can be cast in the following form:

$$\begin{aligned} \mathcal{L}_g = & -i\sqrt{-g} \left[\bar{\psi}_\mu (\not{\nabla} - m) \psi^\mu + \frac{3}{2} \bar{\chi} (\not{\nabla} - m) \chi \right] \\ & - \frac{i}{2m} \sqrt{-g} \left[G^{\mu\nu} (\bar{\chi} \gamma_\mu \psi_\nu - \bar{\psi}_\mu \gamma_\nu \chi) - \frac{1}{2} \bar{\chi} R \chi - \frac{1}{m} G^{\mu\nu} \bar{\chi} \gamma_\mu \nabla_\nu \chi \right]. \end{aligned} \quad (56)$$

With the rescaling $\chi \rightarrow \sqrt{\frac{2}{d-1}} \chi$, the kinetic-like operators for χ become

$$\mathcal{L}_{\chi, \text{kin}} = -i\sqrt{-g} \left[g^{\mu\nu} - \frac{1}{(d-1)m^2} G^{\mu\nu} \right] \bar{\chi} \gamma_\mu \nabla_\nu \chi. \quad (57)$$

The above expression actually holds good even when d is arbitrary. It is clear that, if $G^{\mu\nu}$ is proportional to $g^{\mu\nu}$, the system reduces to a manifestly hyperbolic and causal one. We must ensure, however, that the coefficient in front of $(\bar{\chi} \not{\nabla} \chi)$ is always non-negative. Otherwise, as χ becomes a propagating ghost, there will be loss of unitarity. The coefficient can be computed by noting that, for Einstein spaces one has

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = - \left(\frac{d-2}{2d} \right) g^{\mu\nu} R, \quad (58)$$

which enables one to rewrite (57) as

$$\mathcal{L}_{\chi, \text{kin}} = -i\sqrt{-g} \left[1 + \frac{d-2}{2d(d-1)} \left(\frac{R}{m^2} \right) \right] \bar{\chi} \not{\nabla} \chi. \quad (59)$$

Therefore, everywhere in spacetime, the Ricci scalar must satisfy

$$\left(\frac{d-2}{2d} \right) R \geq -(d-1)m^2. \quad (60)$$

Of special interest are constant curvature spaces, for which the left hand side of (60) is nothing but the cosmological constant Λ . The unitarity bound then reduces to

$$\Lambda \geq -(d-1)m^2. \quad (61)$$

This is precisely the result of [16] for a neutral massive spin 3/2 in cosmological backgrounds. The equality sign in (61) renders the field χ algebraic by setting to zero its kinetic term, and this corresponds to a genuinely massless spin 3/2 field in AdS [13, 16].

3.4 Supergravity

It is well known that $\mathcal{N} = 2$ gauged supergravity [8] incorporates a consistently propagating Rarita-Schwinger field (gravitino), which is coupled to a $U(1)$ field (graviphoton) as well as gravity with cosmological constant. When the cosmological constant is detuned from its supersymmetric value, $\Lambda = -3m^2$, the resulting broken supergravity theory [9, 10] still propagates the massive gravitino causally, for any unitarily allowed Λ , provided the usual mass-charge relation holds [10], i.e. the gravitino charge e under the graviphoton is

$$e = \frac{1}{\sqrt{2}} \left(\frac{m}{M_{\text{P}}} \right). \quad (62)$$

We consider the gravitino as a probe in the dynamical Maxwell-Einstein background; the latter satisfies the bosonic equations of motion of $\mathcal{N} = 2$ (broken) supergravity:

$$\nabla_\mu F^{\mu\nu} = 0, \quad G^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{1}{M_{\text{P}}^2} T^{\mu\nu}, \quad (63)$$

where $T^{\mu\nu}$ is the stress-energy tensor of the Maxwell field, given by

$$T^{\mu\nu} = -\frac{1}{2} F^{+\mu\rho} F_\rho^{-\nu} = -\frac{1}{2} F^{-\mu\rho} F_\rho^{+\nu} = -\left[F^{\mu\rho} F_\rho^\nu - \frac{1}{4} \eta^{\mu\nu} \text{Tr}(F^2) \right]. \quad (64)$$

In this combined background of EM and gravitational fields, the probe Rarita-Schwinger field is described in the unitary gauge by the following non-minimal Lagrangian:

$$\mathcal{L}_{\text{gravitino}} = -i\sqrt{-g} \left[\bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho + m \bar{\psi}_\mu f^{\mu\nu} \psi_\nu \right], \quad f^{\mu\nu} \equiv \gamma^{\mu\nu} + i(e/m^2) F^{\mu\nu}. \quad (65)$$

The commutator of covariant derivatives is given by

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = [\nabla_\mu, \nabla_\nu] + ie F_{\mu\nu}, \quad (66)$$

which, along with the relations (21)-(22), enables one to work out the Stückelberg invariant Lagrangian. Thanks to the Bianchi identities and (24), the result is

$$\begin{aligned} \mathcal{L}_{\text{gravitino}} = & -i\sqrt{-g} \left(\bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho + m \bar{\psi}_\mu f^{\mu\nu} \psi_\nu \right) + i\sqrt{-g} \left(\bar{\psi}_\mu f^{\mu\nu} \mathcal{D}_\nu \chi + \bar{\chi} \overleftrightarrow{\mathcal{D}}_\mu f^{\mu\nu} \psi_\nu \right) \\ & + \frac{e}{2m} \sqrt{-g} \left[F_{\mu\nu} \left(\bar{\chi} \gamma^{\mu\nu\rho} \psi_\rho - \bar{\psi}_\rho \gamma^{\mu\nu\rho} \chi - \bar{\chi} f^{\mu\nu} \chi \right) - \frac{1}{m} F_{\mu\nu} \bar{\chi} \gamma^{\mu\nu\rho} \mathcal{D}_\rho \chi \right] \\ & - \frac{i}{2m} \sqrt{-g} \left[G^{\mu\nu} \left(\bar{\chi} \gamma_\mu \psi_\nu - \bar{\psi}_\nu \gamma_\mu \chi \right) + \frac{1}{2} \bar{\chi} R \chi - \frac{1}{m} G^{\mu\nu} \bar{\chi} \gamma_\mu \mathcal{D}_\nu \chi \right]. \end{aligned} \quad (67)$$

The field redefinition that will remove the kinetic mixing is

$$\psi_\mu \rightarrow \psi_\mu + \frac{1}{2} \left[\gamma_\mu - \frac{i}{2} (e/m^2) \not{F} \gamma_\mu \right] \chi, \quad (68)$$

which can simply be found upon comparison with the model in Section 3.2. The gauge-fixing term to be added is the appropriate version of (11). Thus we are left with

$$\begin{aligned}
\mathcal{L}_{\text{gravitino}} = & -i\sqrt{-g} \left[\bar{\psi}_\mu (\mathcal{D} - m) \psi^\mu + \frac{3}{2} \bar{\chi} (\mathcal{D} - m) \chi \right] \\
& + \frac{e}{2m} \sqrt{-g} \left[F_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu\rho} \psi_\rho - \bar{\psi}_\rho \gamma^{\mu\nu\rho} \chi + \bar{\chi} f^{\mu\nu} \chi) - \frac{1}{m} F_{\mu\nu} \bar{\chi} \gamma^{\mu\nu\rho} \mathcal{D}_\rho \chi \right] \\
& - \frac{i}{2m} \sqrt{-g} \left[G^{\mu\nu} (\bar{\chi} \gamma_\mu \psi_\nu - \bar{\psi}_\mu \gamma_\nu \chi) - \frac{1}{2} \bar{\chi} R \chi - \frac{1}{m} G^{\mu\nu} \bar{\chi} \gamma_\mu \mathcal{D}_\nu \chi \right] \\
& + \frac{i}{2} \sqrt{-g} \chi \left[\left(\gamma_{\mu\nu} + \frac{ie}{m^2} F_{\mu\nu}^+ \right) \left(\gamma^\mu - \frac{ie}{2m^2} \not{F} \gamma^\mu \right) + 3\gamma_\nu \right] \mathcal{D}^\nu \chi. \tag{69}
\end{aligned}$$

The $\mathcal{O}(F)$ contributions coming from the last line exactly cancel the original offending operator $F_{\mu\nu} \bar{\chi} \gamma^{\mu\nu\rho} \mathcal{D}_\rho \chi$, and this can be seen by making use of identities like (50). Then, upon the rescaling $\chi \rightarrow \sqrt{\frac{2}{3}} \chi$, the kinetic-like operators for χ reduce to:

$$\mathcal{L}_{\chi, \text{kin}} = -i\sqrt{-g} \bar{\chi} \left[g^{\mu\nu} - \frac{1}{3m^2} \left(G^{\mu\nu} + \frac{e^2}{m^2} F^{+\mu\rho} F_\rho^{-\nu} \right) \right] \gamma_\mu \mathcal{D}_\nu \chi. \tag{70}$$

Now one can use the equations of motion (63) of the background fields, and the definition (64) of the EM stress-energy tensor $T^{\mu\nu}$, to rewrite the above expression as

$$\mathcal{L}_{\chi, \text{kin}} = -i\sqrt{-g} \bar{\chi} \left[\left(1 + \frac{\Lambda}{3m^2} \right) g^{\mu\nu} - \frac{1}{3m^2} \left(\frac{1}{M_{\text{P}}^2} - \frac{2e^2}{m^2} \right) T^{\mu\nu} \right] \gamma_\mu \mathcal{D}_\nu \chi. \tag{71}$$

If the symmetric tensor inside the brackets is proportional to the metric with a non-negative coefficient, the χ -sector will be ghost-free, and manifestly hyperbolic and causal. This is possible if the factor in front of the stress-energy tensor is set to zero, which is nothing but imposing the charge-mass relation (62). Then, unitarity requires that the cosmological constant be bounded from below: $\Lambda \geq -3m^2$. In this unitarily allowed region, any value of Λ will be consistent, and in particular one can set $\Lambda = 0$.

This shows that the various parameters in $\mathcal{N} = 2$ (broken) supergravity [8, 9, 10] are tuned precisely in a way that ensures a pathology-free helicity-1/2 sector. When $m^2 = -\Lambda/3 = 2e^2 M_{\text{P}}^2$, the kinetic term (71) vanishes, so that χ is relegated to a non-dynamical field. Thus we recover the unbroken $\mathcal{N} = 2$ AdS supergravity [8], where the Rarita-Schwinger field is truly massless and enjoys null propagation.

Notice that arriving at Eq. (71) from Eq. (70) is a non-trivial step, and it crucially depends on the fact that both EM and gravity are dynamical, so that the Einstein equation is sourced by the Maxwell stress-energy tensor. This relates the two a priori different non-canonical kinetic terms in (70), and reduce their number to one. Then the charge-mass relation removes the sole dangerous kinetic-like operator in (71). Finally, one forbids propagating ghosts in the χ -sector by restricting the cosmological constant.

4 Remarks

The purpose of this paper was to demonstrate the power of Stückelberg formalism in making transparent the intricacies associated with interacting theories of a massive Rarita-Schwinger field. All the peculiarities, such as onset of strong coupling, loss of (causal) propagation and unitarity, etc., are essentially captured in the dynamics of the helicity-1/2 mode, and a study thereof elucidates why (in)consistent models are (in)consistent.

We have seen that EM or gravitational interactions of a massive spin 3/2 can have local effective field theory description up to energy scales parametrically larger than the mass. The finite UV cutoff signals the onset of a dynamical regime where the helicity-1/2 sector becomes strongly coupled. Causal propagation does call for non-minimal interactions, which actually lower the intrinsic cutoff of the theory from the theoretical upper bound reported in this paper. For example, in case of EM coupling the required unitary-gauge Pauli term, $i(e/m)\bar{\psi}_\mu F^{+\mu\nu}\psi_\nu$, gives rise to an $\mathcal{O}(e)$ dimension-7 operator in the helicity-1/2 sector, and this lowers the UV cutoff to the scale: $m/\sqrt[3]{e} \ll m/\sqrt{e}$.

We have performed Stückelberg analysis as a consistency check of a number of interacting theories known in the literature. The Velo-Zwanziger acausality [2], of a massive spin 3/2 minimally coupled to EM, indeed shows up as a pathology of the helicity-1/2 mode itself. “Appropriate” non-minimal EM interactions [7] are precisely those that ensure light-cone propagation of this mode. In case of minimal gravitational coupling, the non-canonical kinetic terms can be rendered harmless by requiring spacetime to be an Einstein manifold, provided that the curvature has the well-known unitarity bound; this reconfirms the results of [13, 16]. Finally, we have analyzed $\mathcal{N} = 2$ (broken) supergravity [8, 9, 10] to reveal that the helicity-1/2 sector acquires healthy kinetic terms in the presence of dynamical Maxwell-Einstein fields, if the usual charge-mass relation holds.

The Stückelberg mode(s) can be used as a probe of consistent of interactions for any massive particle with $s \geq 1$. While spin 2 was considered in [11, 12], it remains to be seen what more we can learn from them about consistent interactions of massive higher spins.

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